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A Vortex Method Induced from Two-Dimensional Navier-Stokes Equations

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The purpose of the present paper is to derive an integral equation of Fredholm type with respect to vorticity from the Navier-Stokes equations, to derive a vortex method which is based on the core spreading model, and to analyze the vortex method for viscous fluid flow by using the integral equation. The vortex method consists of two time steps for a simulation cycle; Lagrangian convection simulation for the first step and diffusion simulation derived by quadrature integral formula for the second time step. In the present paper, the governing integral expression with respect to vorticity is derived from two-dimensional Navier-Stokes equations, and the mathematical foundation of vortex method, convergence and stability properties for high Reynolds number, is analyzed under the assumption of smooth initial condition with bounded support and a free-space boundary.

1 Introduction

Vortex methods have been shown to be an attractive and successful approach for the numerical simulation of incompressible fluid flow at high Reynolds number (see Sarpkaya [1]). They have several distinctive advantages as pointed out by Beale and Majda [2]: (1) The physical mechanisms in actual complicated fluid flow can be simulated by the interactions of computational vortices, (2) vortex methods are automatically adaptive, since vortices concentrate in the region of physical interest, and (3) there are no inherent errors such like the numerical viscosity. The mathematical analysis of accuracy and convergence of vortex methods for inviscid fluid flow have been carried out by Hald [3], Beale and Majda [2], Anderson and Greengard [4], and Cottet, et al. [5]. The numerical simulation of viscous fluid flow is based on the fractional method (viscous splitting algorithms): The inviscid Euler equations are solved by discrete vortex methods for the first step and the effects of small viscosity are simulated by a random walk or core spreading for the second step. The theoretical background of the fractional method is cleared by Beale and Majda [6]: The viscous splitting algorithms converge to solutions of the Navier-Stokes equations as the time step approaches to zero. The concepts of random walk and vortex blob are proposed by Chorin [7] to simulate viscous fluid flow, and the vortex blob method combined with the random walk is applied to various flow problems. The theoretical analysis of the random vortex blob method for two-dimensional fluid flow with a free- space boundary is carried out by Long [8] and Goodman [9]. Roberts [10] studied numerically this method in detail and compared with theoretical results. Another approach to simulate viscous fluid flow is the Gaussian core spreading which is based on the solution of heat equation.

The theoretical study of the Gaussian core spreading method was carried out by Greengard [11]: This core spreading algorithm converges to a system of equations different from the Navier-Stokes equations. Cottet et al. [5], on the other hand, proposed an alternative core spreading approach: The weights of vortex particles are changed at each time step without changing their positions such that the conservation property of vorticity is satisfied, and he obtained the stability condition of this method [5]. A new algorithm for the Gaussian core spreading method is proposed by Lu and Ross [12]: Vortex particles are redistributed to the mesh points at each time step. This approach, therefore, is not free for the grid generation.

The present paper aims to analyze the core spreading method, in order to show the theoretical background, by a different mathematical approach from one used by Cottet et al. [5]. First we derive an integral equation of Fredholm type with respect to vorticity and we show: (1) The first iteration of this integral equation is the Gaussian core spreading method, (2) the vorticity decays rapidly with distance from the bounded support of initial vorticity, and (3) the consistency error of this core spreading method is almost of order of time. We prove that this method is indeed stable and consequently convergent up to some time, provided that the vortex field is redistributed at each time step. The present analysis is carried out by maximum norm and the proof of the main theory (Theory 4) will be only shown in this paper.

2 Governing Integral Equation

We here consider a two-dimensional incompressible fluid flow with a free-space boundary, and we take the Cartesian coordinates as (x_1, x_2). Then we have the following vorticity equations from two-dimensional Navier-Stokes equations and the continuity equation:

$$\partial\omega/\partial t + u_i \partial\omega/\partial x_i = \nu\nabla^2\omega \tag{1}$$

$$\omega = -\nabla^2\psi \tag{2}$$

where ω is the vorticity, $u(x,t)(=(u_1,u_2))$ the velocity vector at point $x(=(x_1,x_2))$ and at time t, ψ the stream function, ∇ the nabla operator, and ν the kinematic viscosity. From Eq.(2), the velocity vector $u(x,t)$ at x and at t is obtained by

$$u(x,t) = \int_D K(x-x')\omega(x',t)dx' \tag{3}$$

where $dx' = dx_1' dx_2'$, the integral area D is the whole region of existing ω and the kernel function $K(x)$ is given by

$$K(x) = \frac{1}{2\pi}\frac{(-x_2, x_1)}{|x|^2} \tag{4}$$

Fluid particle is supposed to be approximately convected by a velocity $\hat{u}(x,t)$, which is in $\mathbf{C}^2$: $\mathbf{R}^2 \times \mathbf{R}^+ \to \mathbf{R}^2$ and div$\hat{u} = 0$. Then the approximate trajectory of the fluid particle, $\tilde{x}$, is related with differential equation:

$$\tilde{x} = \Phi_t(a) \tag{5}$$

$$d\tilde{x}/dt = \hat{u}(\tilde{x}, t) \qquad \tilde{x} = a \quad \text{at} \quad t = 0 \tag{6}$$

where $\Phi_t(a)$ is the flow with the initial position a in the velocity field $\hat{u}(x,t)$. We introduce new coordinate system $X(= (X_1, X_2))$ fixed with the approximate trajectory of the vortex particle:

$$X = x - \Phi_t(a) \tag{7}$$

For this new coordinate system, Eq.(1) is expressed as

$$\partial\omega/\partial t = \nu\partial^2\omega/\partial X_i \partial X_i + f(X, a, t) \tag{8}$$

where $\omega = \omega(X, a, t)$, and the function f is the corrected term due to the difference between the exact velocity u and the approximate one $\hat{u}$:

$$\begin{aligned} f(X,a,t) &= (\hat{u}_i(a,t) - u_i(X + \Phi_t(a), t))\partial\omega(X,a,t)/\partial X_i \\ &= \frac{\partial}{\partial X_i}\Big((\hat{u}_i(a,t) - u_i(X + \Phi_t(a))\omega(X,a,t)\Big) \end{aligned}$$

We suppose from Eq.(8) that the solution of Eq.(1) can be expressed as the following integral form:

$$\omega(x,t) = \int_{S_o} \omega_o(a) G(t, |x - \Phi_t(a)|) da + \int_{S_o} da \int_0^t d\tau \int_D \hat{f}(X', a, \tau) G(t - \tau, |x - \Phi_t(a) - X'|) dX' \tag{9}$$

The Green's function $G(t, |x|)$ is the solution of heat equation:

$$G(t, |x|) = \frac{1}{\pi}\frac{1}{\varepsilon_o^2 t}\exp\left(-\frac{|x|^2}{\varepsilon_o^2 t}\right) \tag{10}$$

where $\varepsilon_o = (4\nu)^{1/2}$. Initial vorticity has support S_o inside a bounded region. Since $\hat{f}$ is, at this stage, unknown, we try to derive the governing relation of $\hat{f}$. Using definition of $G(t, |x|)$ and substituting Eq.(9) into the governing equation (1), we finally arrive at

$$\begin{aligned} \int_{S_o} \hat{f}(X,a,t) da &= \frac{\partial}{\partial x_i}\Big(\int_{S_o} \omega_o(a)\hat{u}_i(a,t) G(t, |x - \Phi_t(a)|) da \\ &+ \int_{S_o} da \int_0^t d\tau \int_D \hat{f}(X', a, \tau)\hat{u}_i(a,t) G(t - \tau, |x - \Phi_t(a) - X'|) dX'\Big) \end{aligned}$$

$$
\begin{aligned}
- & \quad u_i \frac{\partial}{\partial x_i}\Big(\int_{S_o} \omega_o(a) G(t, |x - \Phi_t(a)|) da \\
+ & \quad \int_{S_o} da \int_0^t d\tau \int_D \hat{f}(X', a, \tau) G(t - \tau, |x - \Phi_t(a) - X'|) dX'\Big) \qquad (11)
\end{aligned}
$$

Changing the derivative of x_i to X_i and taking into account that S_o is arbitrary, we obtain the integral equation of Fredholm type with respect to $\hat{f}$:

$$
\begin{aligned}
\hat{f}(X, a, t) &= \frac{\partial}{\partial X_i}\Big(\omega_o(a) G(t, |X|)(\hat{u}_i(a, t) - u_i(X + \Phi_t(a), t))\Big) \\
&+ \frac{\partial}{\partial X_i}\Big((\hat{u}_i(a, t) - u_i(X + \Phi_t(a), t)) \\
&\times \int_0^t d\tau \int_D \hat{f}(X', a, \tau) G(t - \tau, |X - X'|) dX'\Big) \qquad (12)
\end{aligned}
$$

We here define a scalar function Ω by

$$
\hat{f}(X, a, t) = \mathrm{div}((\hat{u}(a, t) - u(X + \Phi_t(a), t))\Omega(X, a, t)) \qquad (13)
$$

Then we easily arrive at an alternative expression of Eq.(12):

$$
\begin{aligned}
\Omega(X, a, t) &= \omega_o(a) G(t, |X|) + \int_0^t d\tau \int_D \frac{\partial}{\partial X_i'}\Big((\hat{u}_i(a, t) - u_i(X' + \Phi_\tau(a), \tau)) \\
&\times \Omega(X', a, \tau)\Big) G(t - \tau, |X - X'|) dX' \qquad (14)
\end{aligned}
$$

Remark: Comparing Eq.(14) with Eq.(9), we see;

$$
\omega(x, t) = \int_{S_o} \Omega(x - \Phi_t(a), a, t) da
$$

3 Vortex Method

We suppose that vorticity distribution $\omega(x, t)$ at moment t is known. The core spreading method is that the vorticity field after a lapse of time $\triangle t$ is given by the first term of the right hand side of Eq.(9):

$$
\omega(x, t + \triangle t) = \int \omega(a, t) G(\triangle t, |x - \Phi_{\triangle t}(a)|) da \qquad (15)
$$

The vorticity field is discretized from the above equation by

$$
\tilde{\omega}(x, t + \triangle t) = \sum_j \tilde{\omega}_j G(\triangle t, |x - \tilde{x}_j(t)|) h^2 \qquad (16)
$$

where $\tilde{\omega}_j$ is the vorticity on the jth grid: The set Λ of square grids with squares of side h covers the vorticity distribution $\omega(x, t)$ at moment t. The trajectory $\tilde{x}_j(t + \triangle t)$ of the fluid particle at a certain point in the jth grid at $t + \triangle t$ is calculated from the ordinary differential equation:

$$\frac{d\tilde{x}_j}{dt} = \tilde{u}(\tilde{x}_j, t) \tag{17}$$

$$\tilde{u}(\tilde{x}_j, t) = \int K(\tilde{x}_j - x^{'})\tilde{\omega}(x^{'}, t)dx^{'} \tag{18}$$

In the classical Gaussian core spreading method, the trajectory $\tilde{x}_j$ is calculated by Eqs.(17) and (18), and the vorticity is discretized such like Eq.(16), then the value of $\tilde{\omega}_j$ is the same at every steps. Greengard [11] pointed out that this classical Gaussian core spreading method is not reasonable. Lu and Ross [12] proposed an alternative Gaussian core spreading method: The vortices are rearranged to mesh points at each time step. Cottet et al. [5] also proposed: The weight of the vorticity of the fluid particles is changed at each time step without changing their position.

Algorithm of vortex method can be expressed as

$$\hat{u}^n(x, t) = [\mathbf{A}(\triangle t)]^n u_o(a) \quad \text{at} \quad t = n\triangle t \tag{19}$$

where $\mathbf{A}$ is an one-step operator, $u_o(a)$ is the initial velocity field, and $\triangle t$ is the time step. The one-step operator $\mathbf{A}$ is: The velocity field $\hat{u}(x, t)$ for $(n-1)\triangle t < t \leq n\triangle t$ is obtained for a first step from the given vorticity distribution at $t = (n-1)\triangle t$ by Eq.(18), the fluid particles are converted by Eq.(17), and the vorticity field at $n\triangle t$ is obtained for the second step from Eq.(16), such like Lu and Ross [12] and Cottet et al. [5].

4 Analysis

We consider scalar and vector-valued functions defined on $\mathbf{R}^2$. The class $\mathbf{C}^\lambda(\mathbf{R}^2)$ is defined: Function in class $\mathbf{C}(\mathbf{R}^2)$ is uniformly Hölder continuous on $\mathbf{R}^2$ with exponent λ $(0 < \lambda)$. We assume in this analysis:

Assumption 1: The vorticity ω is in $\mathbf{C}^\lambda(\mathbf{R}^2) \cap \mathbf{L}_1(\mathbf{R}^2)$ for each t in $(0, T]$ for any time T. Further, it decays rapidly for $|x| \to \infty$, such that

$$|\omega(x, t)| \simeq O(G(t, |x|)) \quad \text{for} \quad |x| \to \infty$$

Remark: The velocity field u is given by Eq.(3). Then McGrath [13] shows that $|u| = O(1/|x|)$ as $|x| \to \infty$, if $\omega(x, t)$ is in $\mathbf{C}^\lambda(\mathbf{R}^2) \cap \mathbf{L}_1(\mathbf{R}^2)$ for t in $(0, T]$ for any time T.

Assumption 2: The initial vorticity distribution $\omega_o(a)$ has a bounded support S_o and is in $\mathbf{C}^\lambda(S_o) \cap \mathbf{L}_1(\mathbf{R}^2)$.

From these assumptions, we have from McGrath [13] that for some $\gamma(>0)$

$$|u(x,t) - u(x+\triangle x, t)| \leq M_o(1+T)|\triangle x|^\gamma \tag{20}$$

where M_o is constant independent of $\triangle x$ and t for $t \epsilon (0,T]$.

Assumption 3: The trajectory of the fluid particle, $\Phi_t(a)$, is invertible.

This assumption may be reasonable: If $\Phi_t(a)$ is the exact trajectory of the fluid particle, then in the two-dimensional flow $\Phi_t(a)$ is invertible, because the fluid particle does not arrive at the same position at same moment.

We define a new function Ω_o by

$$\Omega_o(X,a,t) = \Omega(X,a,t)/G(t,|X|) \tag{21}$$

From Assumption 1, we see that $|\Omega_o| \simeq O(1)$ for $|x| \to \infty$. Then, we arrive at

Theorem 1: For any X and $0 < t < T$, there exists a constant C_o independent of a and t such that for some constant α_o independent of a, t, and ε_o,

$$|\Omega_o - \Omega_o^{(0)}| \leq C_o \exp(\alpha_o t|X|/\varepsilon_o)$$

where $\Omega_o^{(0)} = \omega_o$, and $C_o \leq 2|\hat{u} - u|_{\max}$. Further, we have

$$|\Omega_o - \Omega_o^{(0)}| \to O(t^{1/2}) \quad \text{as } t \to 0$$

Remark: We see from this theorem that $\omega(x,t)$ becomes rapidly zero as $|x| \to \infty$, that is, Assumption 1 is satisfied.
Remark: This theorem shows that $\Omega_o - \Omega_o^{(0)}$ is not singular for $t \to 0$. We hence see as $t \to 0$;

$$\begin{aligned} \omega(x,t) &\to \int_{S_o} (\Omega_o - \omega_o(a)) G(t, |x - \Phi_t|) da + \int_{S_o} \omega_o(a) G(t, |x - \Phi_t|) da \\ &\to \omega_o(x,0) \end{aligned}$$

4.1 Consistency of Vortex Method

We denotes the approximate vorticity as

$$\hat{\omega}(x,t) = \int_{S_o} \omega_o(a) G(t, |x - \Phi_t(a)|) da \tag{22}$$

The approximate velocity $\hat{u}(x,t)$ is, therefore, given from Eq.(3) by

$$\hat{u}(x,t) = \int_D K(x - x')\hat{\omega}(x',t)dx' \tag{23}$$

The difference between the exact velocity vector $u(x,t)$ and the approximate one $\hat{u}(x,t)$ becomes from Eqs.(3), (12), and (23) as

$$u(x,t)-\hat{u}(x,t)=\int_{S_o}da\int_0^t d\tau\int_D \hat{f}(X',a,\tau)K(x-x')*G(t-\tau,|x'-\Phi_t(a)-X'|)dX' \quad (24)$$

where

$$\begin{aligned} K(x-x')*G(t,|x'-X|) &\equiv \int_D K(x-x')G(t,|X-x'|)dx' \\ &= \frac{1}{2\pi}\frac{(-x_2+X_2,x_1-X_1)}{|x-X|^2}\left(1-\exp\left(-\frac{|x-X|^2}{\varepsilon_o^2 t}\right)\right) \quad (25) \end{aligned}$$

Here we change the Lagrangian coordinate a to X by $a=\Phi_t^{-1}(x-X)$ for $x-X\epsilon S_t=\Phi_t(S_o)$, where Φ_t^{-1} is the inverse function of Φ_t, then we have from area-preserving and Eq.(24);

$$u(x,t)-\hat{u}(x,t)=\int_S dX\int_0^t d\tau\int_D \hat{f}(X',\Phi_t^{-1}(x-X),\tau)K(x-x')*G(t-\tau,|x'-x+X-X'|)dX' \quad (26)$$

where $S=[X|x-X\epsilon S_t]$.

We define the norm by maximum norm. Then we have

Lemma 1. Suppose that Assumptions 1-3 are satisfied. The velocity difference is given by

$$\begin{aligned} |\hat{u}(x,t)-u(x,t)| &\leq \|\omega_o\|[\|\hat{u}-u\|tC_u(0,0,t)+M_o(1+T)t^{1+\gamma/2}C_u(\gamma,0,t)\varepsilon_o^\gamma] \\ &+ C_o[\|\hat{u}-u\|tC_u(0,\alpha_o,t)+M_o(1+T)t^{1+\gamma/2}C_u(\gamma,\alpha_o,t)\varepsilon_o^\gamma] \\ &\quad \text{for} \quad |x|\leq R_\infty \end{aligned}$$

where R_∞ is some large value independent of ε_o such that for any small $\delta(>0)$,

$$|\hat{u}(x,t)-u(x,t)|\leq O(\delta) \quad \text{for} \quad |x|>R_\infty$$

The constant, $C_u(\gamma,\alpha_o,t)$, is given by

$$\begin{aligned} C_u(\gamma,\alpha_o,t) &= \pi^{1/2}2^{-\gamma}\frac{\Gamma(\gamma+2)}{\Gamma((\gamma+3)/2)}\Big[\Big(1+\int_0^1\frac{1-\exp(-x)}{x}dx+2\log(R_t+R_\infty) \\ &- 2\log\varepsilon_o-2\log t\Big)/(1+\frac{\gamma}{2})-2\int_0^1 x^{\gamma/2}\log(1-x)dx\Big]+\pi^{1/2}2^{-\gamma}\frac{\Gamma(\gamma/2+1)}{\Gamma(\gamma+5/2)} \\ &+ \pi^{1/2}2^{-\gamma-1/2}\frac{\varepsilon_o}{R_t+R_\infty}\frac{\Gamma(\gamma+3)}{\Gamma(\gamma/2+2)}\frac{1}{\gamma/2+3/2} \end{aligned}$$

Here, R_t is the ball of S_t. The function, $\Gamma(x)$, is the Gamma function, and C_o is the constant defined by Theorem 1.

Remark: Taking into account of the remark of Assumption 1 that velocity decays for

$|x| \to \infty$, we easily see the existence of R_∞.

From this lemma, we easily have:

Lemma 2. There exists a time t_o such that $\beta_o(t) < 1$ for $0 < t \leq t_o$ and we have for $0 < t \leq t_o$

$$\|\hat{u} - u\| \leq M_o(1+T)\gamma_o(t)t^{1+\gamma/2}\varepsilon_o^\gamma/(1-\beta_o(t))$$

where

$$\begin{aligned} \beta_o(t) &= t[\|\omega_o\|C_u(0,0,t) + C_o C_u(0,\alpha_o,t)] \\ \gamma_o(t) &= \|\omega_o\|C_u(\gamma,0,t) + C_o C_u(\gamma,\alpha_o,t) \end{aligned}$$

Note: Since $C_u(\gamma,\alpha_o,t)$ is function of $-\log t$ with respect to t and $t\log t \to 0$ as $t \to 0$, we see that there exists the time t_o.

Let us apply Lemma 2 to the vortex method, Eq.(19), from $(n-1)$ time step to n step. Then we have

$$\|(\hat{u}^n - \tilde{u}^{n-1}) - (u^n - \tilde{u}^{n-1})\| \leq M_o(1+T)\gamma_o(\triangle t)\triangle t^{1+\gamma/2}\varepsilon_o^\gamma/(1-\beta_o)$$

where $\beta_o = \beta_o(\triangle t)$, $\hat{u}^n = \hat{u}(x, n\triangle t)$, $u^n = u(x, n\triangle t)$, and $\tilde{u}^{n-1}$ is velocity vector which is obtained from the vorticity field given at $t = (n-1)\triangle t$. Therefore, the following relation is easily derived

$$\|\hat{u}^n - u^n\| \leq \sum_{i=1}^{n} \|\hat{u}^i - \tilde{u}^{i-1} - (u^i - \tilde{u}^{i-1})\| \leq M_o(1+T)\gamma_o(\triangle t)\triangle t^{\gamma/2} t\varepsilon_o^\gamma/(1-\beta_o) \quad \text{for } t = n\triangle t$$

From this fact, we arrive at

Theorem 2. Suppose that Assumptions 1-3 are satisfied. Then for small time step $\triangle t$ such that $\beta_o < 1$, the vortex method given by Eq.(19) becomes as

$$\|\hat{u}^n - u^n\| \leq M_o(1+T)\gamma_o(\triangle t)\triangle t^{\gamma/2} t\varepsilon_o^\gamma/(1-\beta_o)$$

where $t = n\triangle t$.
Remark: McGrath [13] shows that $0 < \gamma < 1$ if ω is in $\mathbf{L}_1(\mathbf{R}^2)$ for every $t\epsilon(0,T]$ and uniformly Hölder continuous with exponent γ. Cottet et al. [5] show that the error of velocity is less than $\triangle t\varepsilon_o^2$, under the assumption that ω_o is smooth enough.

4.2 Stability of Vortex Method

To prove the stability of the vortex method, Eq.(19), we consider first the stability of the approximate velocity, and second we will prove the stability of the vortex method. For the stability of the approximate velocity, we have:

Lemma 3. Suppose that Assumptions 1-3 are satisfied. We have

$$|\hat{u}(x,t) - \hat{u}(\tilde{x},t)| \leq |\triangle x| \|\omega_o\| (\hat{C}_{oo} - \log|\triangle x|)$$

where $\triangle x \equiv \tilde{x} - x$, $\hat{C}_{oo}$ is positive constant independent of ε_o and t:

$$\hat{C}_{oo} = \frac{2}{\pi}\Big(\int_0^1 K(x)dx + \int_1^\infty \frac{1}{x}\{K(1/x) - \frac{\pi}{2}\}dx\Big) + \log(R_t + R_\infty) + 3$$

where $K(x)$ is the complete elliptic integral of the first kind.
Remark: McGrath [13] obtained this results already, however, he did not estimate the constant $\hat{C}_{oo}$.

The difference of the approximate velocities at n time step between x and $\tilde{x}$ is given by

$$\begin{aligned} |\hat{u}^n(x) - \hat{u}^n(\tilde{x})| &= |(\hat{u}^n(x) - \tilde{u}^{n-1}(x)) - (\hat{u}^n(\tilde{x}) - \tilde{u}^{n-1}(x))| \\ &\leq \sum_{i=1}^{n} |(\hat{u}^i(x) - \tilde{u}^{i-1}(x)) - (\hat{u}^i(\tilde{x}) - \tilde{u}^{i-1}(x))| \end{aligned}$$

where $\tilde{u}^n(x) = \tilde{u}(x, n\triangle t)$ is defined in section 4.1, and $\hat{u}^n(x) = \hat{u}(x, n\triangle t)$. From this result, we arrive at the stability theorem by using Lemma 3:

Theorem 3. Suppose that Assumptions 1-3 are satisfied. Then we have the following relation for the vortex method, Eq.(19):

$$|\hat{u}(x,t) - \hat{u}(\tilde{x},t)| \leq |\triangle x| \frac{\|\omega_o\|}{\triangle t} t(\hat{C}_{oo} - \log|\triangle x|)$$

Let us denote the discretized vorticity on the ith grid $\mathbf{\Lambda}_i$ by ω_i. Then the error of velocity field due to the discretization of the vorticity field ω is given by

$$e_c = |\sum_i K_\epsilon(x - \tilde{x}_i(t))\omega_i h^2 - \int_D K_\epsilon(x - x')\omega(x', t)dx'| \tag{27}$$

where $K_\epsilon(x) = K(x - x') * G(t, |x'|)$ and $\tilde{x}_i$ is a point in $\mathbf{\Lambda}_i$. Anderson and Greengard [4] and others [2]- [5] show;

$$e_c \leq C_c \left(\frac{h}{\varepsilon_o}\right)^L \varepsilon_o \tag{28}$$

where C_c and $L(3 \leq L < \infty)$ are constant independent of h and ε_o. Using this result, we arrive at the following stability theorem:

Theorem 4. The error, $\triangle x = \tilde{x} - x$, is given by the following relation for $\triangle t$ such that $\beta_o \equiv \beta_o(\triangle t) < 1$:

$$|\triangle x| \leq \frac{7}{4}[1 - \|\omega_o\| C_2 \triangle t \log\{C_2 H(\triangle t)\triangle t\}]H(\triangle t)t$$

where x and $\tilde{x}$ are the exact trajectory of the fluid particle and the approximate one obtained by the vortex method, respectively, C_2 is a constant independent of t, and

$$H(t) = C_c \left(\frac{h}{\varepsilon_o}\right)^L \varepsilon_o + \frac{M_o(1+T)}{1-\beta_o}\gamma_o(t)t^{1+\gamma/2}\varepsilon_o^\gamma$$

Proof. Let an initial vorticity, ω_o, be given. The approximate trajectory of the fluid particle is given by the vortex method;

$$\frac{d\tilde{x}}{dt} = \sum_i K_\epsilon(\tilde{x} - x_i)\omega_{oi}h^2 \tag{29}$$

Then we have

$$\frac{d(\tilde{x} - x)}{dt} = \sum_i K_\epsilon(\tilde{x} - x_i)\omega_{oi}h^2 - \int_D K(x - x')\omega_o(x', t)dx' \tag{30}$$

From Eqs.(28) and (30), we have

$$|\frac{d\triangle x}{dt}| \leq C_c \left(\frac{h}{\varepsilon_o}\right)^L \varepsilon_o + |\hat{u}(x,t) - \hat{u}(\tilde{x},t)| + |\hat{u}(x,t) - u(x,t)| \tag{31}$$

We note that u and $\hat{u}$ are defined by

$$u(x,t) = \int_D K(x - x')\omega_o(x', t)dx' \quad \hat{u}(x,t) = \int_D K_\varepsilon(x - x')\omega_o(x', t)dx'$$

From Lemma 2, we have $\beta_o < 1$ for time $t \leq t_o$:

$$\frac{d|\triangle x|}{dt} \leq H(t) + |\hat{u}(x,t) - \hat{u}(\tilde{x},t)|$$

From Theorem 3, we have

$$\frac{d|\triangle x|}{dt} \leq H(t) + \|\omega_o\||\triangle x|(\hat{C}_{oo} - \log|\triangle x|)$$

Suppose that $0 \leq |\triangle x| \ll 1$, then we have for $0 < t \leq \triangle t$; $H(t) \leq H(\triangle t)$. We therefore have for $\triangle t \leq t_o$:

$$\frac{d|\triangle x|}{dt} \leq H(\triangle t) + \|\omega_o\|\hat{C}_{oo}|\triangle x| - \|\omega_o\||\triangle x|_{max} \log|\triangle x|_{max}$$

where $|\triangle x|_{max}$ is the maximum of $|\triangle x|$. From the Gronwall inequality, we have for $0 < t \leq \triangle t$

$$|\triangle x| \leq \frac{1}{\|\omega_o\|\hat{C}_{oo}}[H(\triangle t) - \|\omega_o\||\triangle x|_{max}\log|\triangle x|_{max}][\exp(\|\omega_o\|\hat{C}_{oo}t) - 1] \qquad (32)$$

From 4.2.38 in [14], we have

$$|\triangle x| < \frac{7}{4}[H(\triangle t) - \|\omega_o\||\triangle x|_{max}\log|\triangle x|_{max}]t \qquad (33)$$

We have from this equation for $0 < t \leq \triangle t$:

$$|\triangle x|_{max} < \frac{7}{4}[H(\triangle t) - \|\omega_o\||\triangle x|_{max}\log|\triangle x|_{max}]\triangle t$$

Thus, we have

$$|\triangle x|_{max} \leq C_2 H(\triangle t)\triangle t \qquad (34)$$

where C_2 is constant with respect to t and $C_2 \sim -\log\triangle t$ as $\triangle t \to 0$. Thus, we have from Eq.(34)

$$|\triangle x| < \frac{7}{4}[H(\triangle t) - \|\omega_o\|C_2 H(\triangle t)\triangle t\log\{C_2 H(\triangle t)\triangle t\}]t \qquad (35)$$

Let us consider the difference between the exact velocity and the approximate one from $(n-1)\triangle t$ to $n\triangle t$. From Eq.(35), we hence have

$$|\triangle x^n - \triangle x^{n-1}| \leq \frac{7}{4}[1 - \|\omega_o\|C_2\triangle t\log\{C_2 H(\triangle t)\triangle t\}]H(\triangle t)\triangle t \qquad (36)$$

where $\triangle x^n = \tilde{x}(n\triangle t) - x(n\triangle t)$. We use the relation;

$$\begin{aligned} |\hat{u}^n - u^n| &= |(\hat{u}^n - \tilde{u}^{n-1}) - (u^n - \tilde{u}^{n-1})| \\ &\leq \sum_i^n |(\hat{u}^i - \tilde{u}^{i-1}) - (u^i - \tilde{u}^{i-1})| \end{aligned}$$

where $\hat{u}^n = \hat{u}(\tilde{x}(n\triangle t), n\triangle t)$ and $u^n = u(x(n\triangle t), n\triangle t)$. $\tilde{u}^{n-1}$ is defined in section 4.1. Then, we easily arrive at this theorem.

Remark: From this theory, the accuracy of the vortex particle due to the vortex method, Eq.(19), is of order of $O((h/\varepsilon_o)^L\varepsilon_o, \varepsilon_o^\gamma\triangle t^{1+\gamma/2})$ for small ε_o and $\triangle t$. We have to take the grid size h for high Reynolds number flows as;

$$h \leq O(\varepsilon_o^{1+(\gamma-1)/L}\triangle t^{(1+\gamma/2)/L}) \qquad (37)$$

Then the effect of the discretization of vorticity field to the numerical error is much smaller than that of the approximation of the core spreading method. Cottet et al. [5] show the stability condition $h \leq (\varepsilon_o^2\triangle t)^{1/2+s}$ for arbitrary small $s > 0$, under the assumption that

ω_o is smooth enough. We note that in the present paper ω_o is in $\mathbf{C}^\lambda$ and Cottet et al. [5] do not show the stability theorem 3.
Remark: In the present analysis, the approximate vorticity field is assumed to be negligible small for $x > R_\infty$ and its effect to the approximate velocity field is assumed to be also negligibly small for each simulation cycle. This assumption may be rational; in the present analysis the initial vorticity has bounded support, and the effect of diffusion due to viscosity becomes small exponentially with $|x|^2$ from Theorem 1.

The present analysis shows the errors of velocity field and fluid particle after a lapse of time $\triangle t$ from same vorticity field. This fact implies that the present results can not extend to the classical core spreading method treated by Greengard [11]. We see from the process of proving Theorem 4 that in the classical core spreading method the difference between the exact trajectory and the approximate one implies to become large with time (see Theorem 3). Thus, the present analysis is applicable to the algorithms of Lu and Ross [12] or Cottet et al. [5]: Vorticity distribution is rearranged at each time step.

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